

IV. Options

- Economic activity, and trading, involves risk.

One may have to make decision based on prediction of the future in the presence of uncertainty.

One may or may not regret having made this choice.

- An *option* is a financial instrument giving one ***the right but not the obligation*** to make a specified transaction at (or by) a specified date at a specified price.

Whether or not the option will be exercised depends on the uncertain future.

1. Types of option

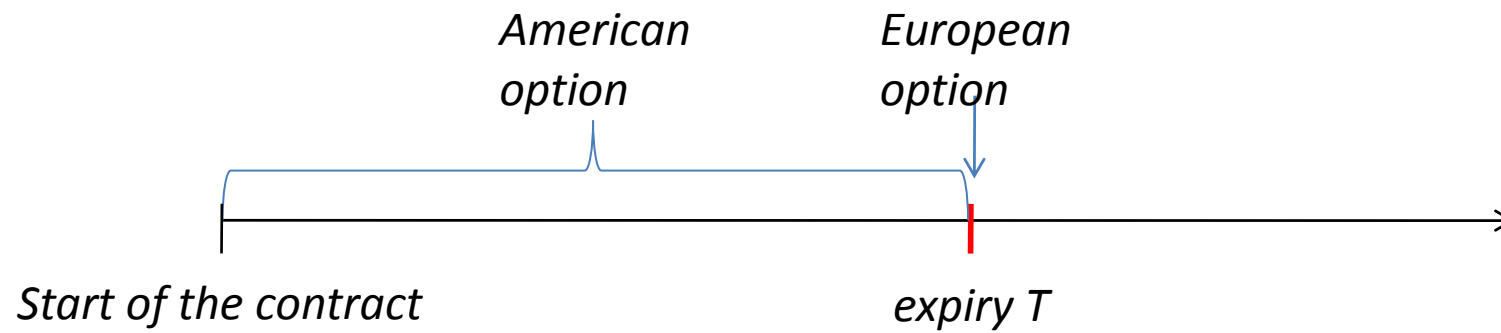
Call options: give one the right (but not the obligation) to *buy*.

Put options: give one the right (but not the obligation) to *sell*.

European options: give one the right to buy/sell on a specified date, the *expiry date*, when the option expires or matures.

American options: give one the right to buy/sell at any time prior to or at expiry.

Exercise time for American and European options:



Note: the terms European, American (Asian, Bermudan, Russian,...) only refer to the type of option – no relation to the area in the name!

Simple call and put options are called *vanilla options*.

More complex options are:

- *Asian* options: depends on the average price over a period.
- *Russian* options: depends on the maximum or minimum price over a period.
- *Barrier* options: depend on some price level being attained or not.

2. Terminology

The asset to which the option refers is called the *underlying* (asset).

The price at which the transaction to buy/sell the underlying on/by expiry date (if exercised) is called the *exercise price* or *strike* (usual notation: K).

Consider a European call option with strike K . Write S_t for the price of the underlying at time t .

If at expiry T : $S_T > K$, the option is *in the money*: the holder should/will exercise the option obtaining an asset worth $S_T (> K)$ for K . He can then immediately sell the asset for S_T making a profit of $S_T - K (> 0)$.

If $S_T = K$, the option is said to be *at the money*.

If $S_T < K$, the option is *out of the money*, and should not be exercised.

Thus, the payoff from the option is:

$$S_T - K \text{ if } S_T > K, \quad 0 \text{ otherwise}$$

Which can be written more briefly as:

$$\boxed{\max(S_T - K, 0)} \text{ or } \boxed{(S_T - K)_+}$$

Similarly, the payoff from a put option is:

$$K - S_T \text{ if } S_T \leq K, \quad 0 \text{ if } S_T > K$$

Which can be written:

$$(K - S_T)_+$$

3. Option pricing

Recall: modern theory began with Black-Scholes formula for pricing European option.

We will deal with that and cover the pricing of European option + discuss American options (harder, no explicit formula hence need to evaluate them numerically).

First an example: single-period binary model

Our risky asset is the current price in Swiss Francs (SFR) of \$100, say this is $X_0 = 150$, *i.e. exchange rate at time $t=0$ is 1.50.*

Consider a call option with strike price $c=150$ at expiry T (*i.e. Option is at the money*).

The simplest case is the binary model with 2 outcomes at time T . Suppose the price of \$100 at time T is (in SFR):

$$X_T \begin{cases} 180, & \text{with probability } p \\ 90, & \text{with probability } 1-p \end{cases}$$

So, the payoff H of the option will be:

$$H = \max(180-150, 0) = 30, \text{ with probability } p$$

$$H = \max(90-150, 0) = 0, \text{ with probability } 1-p$$

So has the expectation: $E[H] = 30p$

This would seem to be the fair price for the option at time $t=0$. Or allowing for an interest-rate r over the period of interest and discounting, we get:

$$V_0 = E\left(\frac{H}{1+r}\right) = \frac{30p}{1+r}.$$

For simplicity, assume $p=0.5$ and $r = 0$ (no interest), the naive or expectation value of the option at time $t=0$ is:

$$V_0 = 15$$

Now the *Black-Scholes value* of the option is different. We derive it we follow the *B-S* prescription as follows:

- i. First replace p by p^* so that the price properly discounted *behaves like a fair game*:

$$X_0 = E^*\left(\frac{X_T}{1+r}\right)$$

that is: $150 = \frac{1}{1+r}(p^*.180 + (1-p^*).90)$

for $r = 0$, this gives $60 = 90p^*$ or $p^* = 2/3$

- ii. Now, compute the fair price of the expected payoff in this “new model”:

$$V_0 = E^*\left(\frac{H}{1+r}\right) = \frac{30p^*}{1+r};$$

For $r = 0$, this gives the B - S value as $V_0 = 20$. (*)

It works!!!

Justification (again for simplicity we take $r = 0$):

We **sell** the option at time 0, for a price $\pi(H)$, say. We then prepare for the future contingent claim on us at time T , by the option holder, by using the following strategy:

- Sell the option for $\pi(H)$,
- Buy \$ 100/3 (= \$33.3333) at the present exchange rate of 1.50: - **50** (this costs us SFR 50 and we have \$33.3333),
- Borrow SFR 30: + **30**,
- Balance at time $t=0$ is : $\pi(H) - 50 + 30 = \pi(H) - 20$.

At time $t=T$, as seen above, two cases are possible:

1. The dollar has risen, *i.e.* \$100 is now worth SFR 180:
Option is exercised (against us): **-30**
Sell dollars at 1.80: **+60**
Repay loan: **-30**
Balance at time $t=T$: **0**
2. The dollar has fallen, *i.e.* \$100 is now worth SFR 90:
Option is not exercised
Sell dollars at 0.90: **+30**
Repay loan: **-30**
Balance at time $t=T$: **0**

Important point: we have found a portfolio which synthesises the cash flow of the option. Holding this portfolio today at time t is equivalent to holding the option – they produce same cash-flow in the future.

The price of the option should be the same as the price of constructing the portfolio.

The balance $\pi(H) - 20$ at time 0 should therefore be 0, giving the *B-S* price $\pi(H) = 20$, as above in (*).

Any other price gives an arbitrage opportunity:

- if the option is offered too cheaply, buy it.
- If it is offered too dearly, *write* it.

Any other price would offer an arbitrageur the opportunity to extract a riskless profit by appropriately buying and selling Swiss Francs, US dollars and options.

Question: How did we find the right amount of \$ to buy and the amount of SFR to borrow for the construction of our portfolio?

Notes:

- Same argument with interest-rate applies (divide everything by $1 + r$).
- This argument and result are **independent of p** the “real” probability and depend instead **only** on this “fictitious” new p^* probability - called the *risk-neutral* or *risk-adjusted* probability.

This example is highly instructive and the theory here is the *Black-Scholes* theory, which we consider below.

The technical key to the *Black-Scholes* prescription is the introduction of p^* and its associated expectation operator E^*

In technical language , this is the *equivalent martingale measure*. Recall equivalent measures and martingales, Chapter 3?

4. Risk Neutral Valuation

We call a market *complete* if all contingent claims (financial claims, such as options, dependent or contingent on uncertain outcomes) can be replicated (are equivalent to a suitable combination of cash and stock).

Recall a market is NA if it has no arbitrage.

The key to mathematical finance is the following:

Theorem 1.

The market is NA iff there exists an equivalent martingale measure (EMM) – a probability measure equivalent to the original one under which discounted asset prices become martingales.

Theorem 2.

The market is complete iff the EMM is unique.

Combining and extending, we have:

Theorem 3 (Fundamental Theorem of Asset Pricing).

- i. The market is NA and complete iff there exists a unique EMM, P^* .
- ii. The (no-arbitrage) price at time $t \in [0, T]$ of a contingent claim C at expiry T is:

$$\pi_C(t) = E^* \left[e^{-r(T-t)} C \mid F_t \right]$$

where E^* is the expectation under the EMM P^* and r is the risk-free interest rate.

These results form the basis of the *arbitrage pricing technique* (APT).

Recall the SDE for **GBM**: the RHS is $\mu dt + \sigma dB_t$.

The change from the original (*real-world*) probability measure P to the EMM P^* changes:

$$\boxed{\mu dt} \xrightarrow{\text{to}} \boxed{r dt}$$

i.e. to change the rate on the risky stock to the riskless return rate.

The role of discounting is to remove the $r dt$, leaving us with σdB_t .

The discounted asset price is then a stochastic (Itô) integral:

$$\int_0^t H_s dB_s$$

here H_t represents how many unit of stocks we should hold at time t .

We should think of $dB_t = B(t + dt) - B(t)$ as the new price-sensitive information jutting out from time t into the future $[t, t+dt]$.

Then $H_t dB_t = H_t(B(t + dt) - B(t))$ represents our gain from trading over $[t, t+dt]$.

Summing these $\int_0^t H_s dB_s$ represents our gain from trading over $[0, t]$.

The EMM that makes the risky return rate μ equal to the riskless return r is called the *risk-neutral measure*.

Theorem 3 ii is also called the *risk-neutral valuation principle*.

A *Brownian martingale* is a martingale w.r.t. the Brownian filtration - the filtration (or information flow) of a Brownian motion.

Quote: All Brownian mgs can be represented as stochastic integrals in this way (*Representation Theorem for Brownian Martingales*).

5. Complete vs. Incomplete Market

Recall **Theorem 2**, the market is complete iff there is a unique EMM. By **Theorem 3** ii the price comes from the EMM, so if this is not unique, the price is not unique!

“Prices are unique iff the market is complete”

When prices are not unique they fill out an interval, we interpret this interval as the *bid-ask spread*.

We know this already - think of when one wishes to change sterling for a foreign currency, the exchange bureau will quote 2 prices:

- one for buying,
- one for selling.

Difference is necessary to cover the overheads of the bureau.

This is reminiscent of the difference between interest rates for borrowing and for lending (Chapter 1).

Real markets are in fact *incomplete*. But incomplete markets are much harder to analyse!!

We therefore restrict ourselves to complete markets for convenience.

6. American options – infinite time horizon

Recall:

American option: one can exercise at any time until expiry

Hence,

American option with infinite time horizon: one can exercise at any time over an infinite time horizon.

Consider a *put* option (see Chapter V for the *call* option). Giving the right to sell at the strike price K , at any time τ of our choosing.

This time τ has to be a *stopping time*: we have to take the decision whether or not to stop at τ based on the information already available – *i.e.* contained in $\mathcal{F}_\tau$, no access to future.

As seen above, we pass to the *risk-neutral measure*, under which the SDE for GBM becomes:

$$dX_t = rX_t dt + \sigma X_t dB_t \quad (\text{GBMr})$$

To evaluate the option, we have to solve the *optimal stopping problem*:

$$V(x) := \sup_{\tau} E_x[e^{-r\tau}(K - X_{\tau})^+]$$

where the supremum is taken over all stopping time τ and $X_0 = x$ under P_x (note that we used the notation E_x).

The process X satisfying (GBMr) is specified by a second-order linear differential operator, called its *infinitesimal generator*,

$$L_X := rxD + \frac{1}{2}\sigma^2 x^2 D^2, \quad D := \partial/\partial x$$

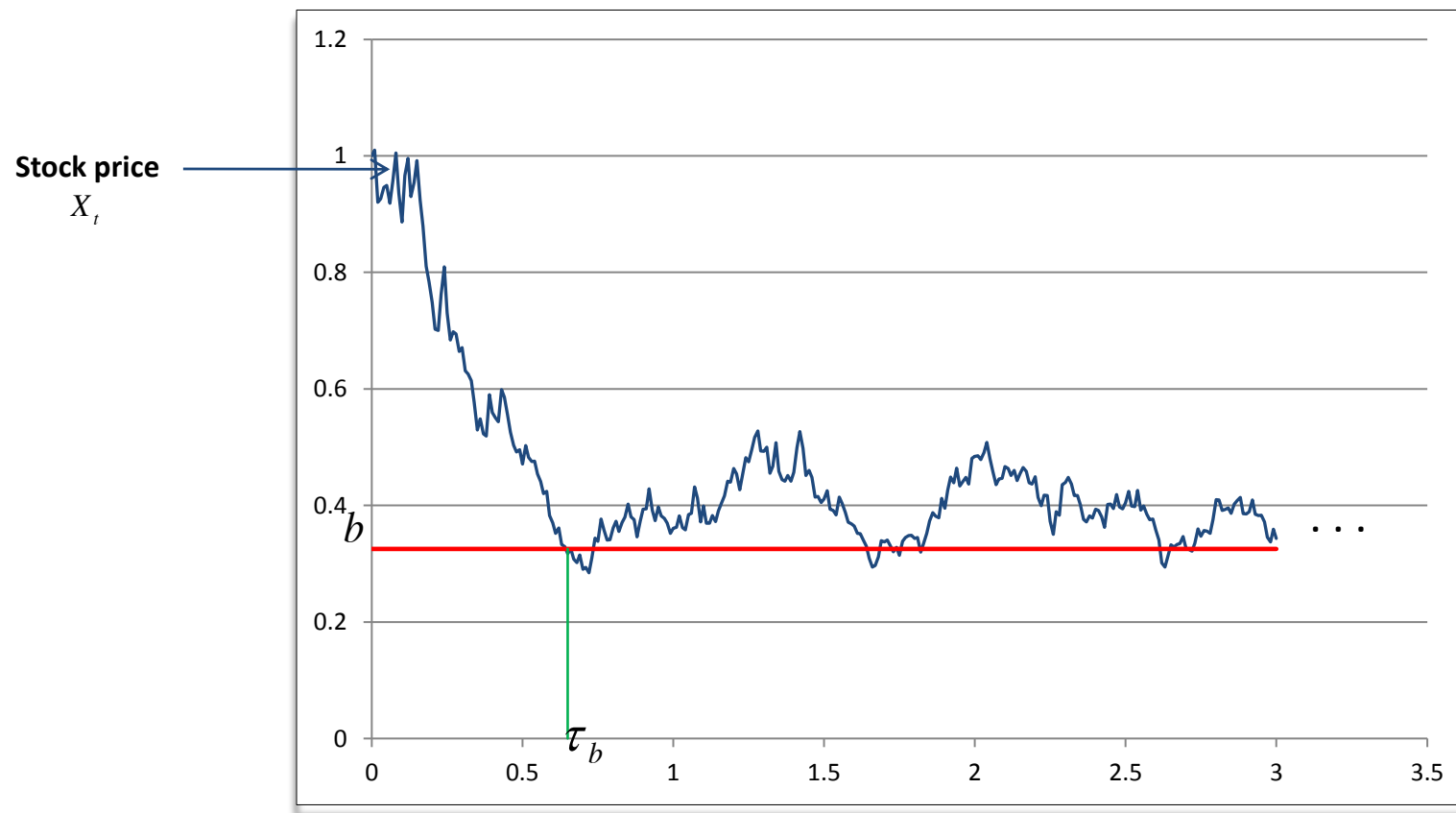
Now, the closer X gets to 0, the less likely we are to gain by continuing.

This suggests that our best strategy is to stop when X gets too small, *i.e.* to stop at $\tau = \tau_b$, where

$$\tau_b := \inf\{t \geq 0 : X_t \leq b\}$$

for some $b \in (0, K)$.

Illustration of first hitting time (τ_b)



This gives the following *free boundary problem* for the unknown value function $V(x)$ and the unknown point b :

$$L_X V = rV \quad \text{for } x > b; \quad (i)$$

$$V(x) = (K - x)^+ \quad \text{for } x = b; \quad (ii)$$

$$V'(x) = -1 \quad \text{for } x = b \text{ (smooth fit);} \quad (iii)$$

$$V(x) > (K - x)^+ \quad \text{for } x > b; \quad (iv)$$

$$V(x) = (K - x)^+ \quad \text{for } 0 < x < b. \quad (v)$$

Writing $d := \sigma^2/2$ (“ d ” for diffusion) , (i) is:

$$dx^2 V'' + rxV' - rV = 0 \quad (i^*)$$

Trial solution: $V(x) = x^p$

Substituting gives a quadratic for p : $p^2 - (1 - \frac{r}{d})p - \frac{r}{d} = 0$

Two roots are: $p = -1$ and $p = -r/d$

So general solution (GS) to (i*) is:

$$V(x) = C_1 x + C_2 x^{-r/d}$$

for some constant C_1 and C_2 .

But we must have $V(x) \leq K$ for all $x \geq 0$ as an option giving the right to sell at price K cannot be worth more than K .

So: $C_1 = 0$.

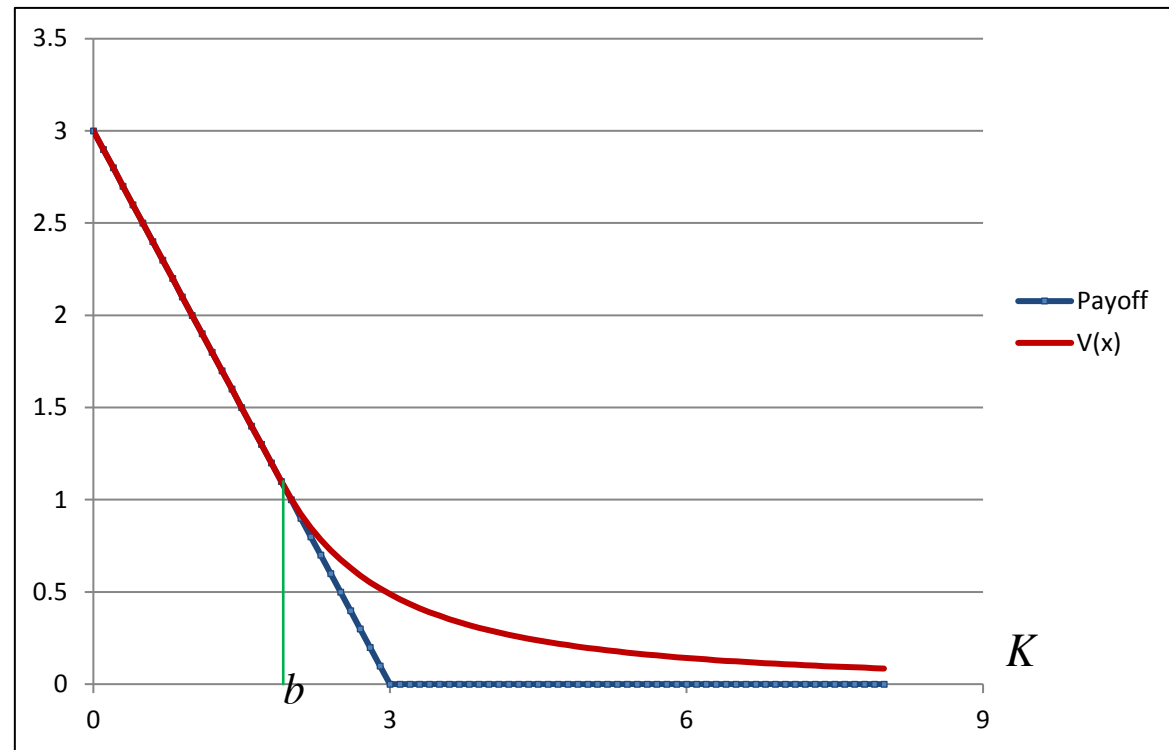
This gives: $C_2 = \frac{d}{r} \left(\frac{K}{1 + d/r} \right)^{1+r/d}, \quad b = \frac{K}{1 + d/r}.$

The full and correct solution to the problem is:

$$\begin{aligned} V(x) &= \frac{d}{r} \left(\frac{K}{1 + d/r} \right)^{1+r/d} x^{-r/d} && \text{if } x \in [b, \infty) \\ &= K - x && \text{if } x \in (0, b]. \end{aligned}$$

For more, see [P&S].

Plotting the solution with $K = 3, \sigma = 15\%, r = 2\%$:



The “*smooth*” fit in (iii) is characteristic of free boundary problems:
no kink!

7. Discrete Black-Scholes formula

Recall the binomial tree model.

Suppose:

- prices go up by a factor of $(1+u)$ with probability p and down by a factor $(1+d)$ with probability $(1-p)$.
- risk-free interest rate per period is r - *i.e.* cash invested goes up by a factor of $1 + r$ per period.

Then there is a unique risk-neutral value of p given by:

$$p^* = (r - d) / (u - d)$$

Choose any terminal node. You can count how many “up” and “down” steps are needed to reach from the root.

Its P^* -probability is then a binomial probability (recall EPM944) of the form:

$$\binom{n}{k} (p^*)^k (1 - p^*)^{n-k} ,$$

where k denotes the number of “up steps” and $n-k$ the number of “down steps”.

So the E^* -expectation of the discounted pay-off at expiry is given by a binomial sum.

This is the *discrete Black-Scholes formula*.

Note:

Recall, by the Central Limit Formula, the binomial distribution approximates a Normal distribution in the limit.

One can carry out such a binomial-to-normal approximation here.

The discrete Black-Scholes formula goes over to the (continuous) Black-Scholes formula (with the use of the standard normal distribution function – see below).

8. Black-Scholes formula

Most famous formula in mathematical finance.

The price process under the original probability P is GBM with parameters

$$\mu, \sigma, GBM(\mu, \sigma)$$

But under the risk-neutral probability P^* is:

$$GBM(r, \sigma)$$

S_t has P^* -dynamics:

$$dS_t = rS_t dt + \sigma S_t dB_t, \quad S_t = s$$

Solving the SDE for GBM as before, we can write S_T explicitly as:

$$S_T = s \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T - t) + \sigma(B_T - B_t)\right\} \quad (**)$$

Now recall that $B_T - B_t$ is normal $N(0, T-t)$ so:

$$(B_T - B_t)/\sqrt{T-t} =: Z \text{ is standard normal.}$$

Rewrite (**) as follows:

$$S_T = s \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T - t) + \sigma Z \sqrt{T-t}\right\}$$

with $Z \sim N(0, 1)$.

So by the Risk-Neutral Valuation Formula, the pricing formula is:

$$F(t, s) = e^{-r(T-t)} \int_{-\infty}^{\infty} \boxed{h}\left(s \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(T-t)^{\frac{1}{2}}x\right\}\right) \cdot \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx$$

(***)

For a general payoff function $h(\cdot)$ there is no explicit formula for the integral (need to evaluate it numerically).

However, we can evaluate the integral analytically for the basic case of a European call option strike K , *i.e.* when:

$$h(s) = (S - K)^+$$

This gives for (***):

$$F(t, s) = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \left[s \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(T-t)^{\frac{1}{2}}x\right\} - K \right]_+ dx$$

(****)

Which is equal to...

Continuous Black-Scholes Formula

$$F(t, s) = s\Phi(d_+) - e^{-r(T-t)}K\Phi(d_-)$$

where (writing Φ for the standard normal distribution function):

$$d_{\pm} := [\log(s/K) + (r \pm \frac{1}{2}\sigma^2)(T - t)]/\sigma\sqrt{T - t}$$

Proof. The integration happens only when $[...] > 0$, hence when:

$$s \exp\{...\} > K$$

Rearranging, $s \exp\{\dots\} > K$ gives:

$$x > -c := \frac{\log(K/s) - (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}},$$

Note that if $-c$ is defined as above, we have:

$$c = \frac{\log(s/K) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$

Hence the second term in (****) gives:

$$\begin{aligned} e^{-r(T-t)} K \int_{-c}^{\infty} \phi(x) \, dx &= e^{-r(T-t)} K [1 - \Phi(-c)] \\ &= e^{-r(T-t)} K \Phi(c), \end{aligned}$$

where the second equality uses the symmetry of the standard normal distribution.

The first term in (****) gives:

$$\begin{aligned}
 & s \int_{-c}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}[x^2 - 2\sigma\sqrt{T-t}.x + \sigma^2(T-t)]\right\} dx \\
 &= s \int_{-c}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x - \sigma\sqrt{T-t})^2\right\} dx \\
 &= s \int_{-c - \sigma\sqrt{T-t}}^{\infty} \phi(u) du \\
 &= s[1 - \Phi(-c - \sigma\sqrt{T-t})] \\
 &= s\Phi(c + \sigma\sqrt{T-t}).
 \end{aligned}$$

Now: $c + \sigma\sqrt{T-t} = \frac{\log(s/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$

and this completes the proof $\square$

Notes on all this:

1. The most surprising feature of the B-S: the real rate μ of the stock price **does not appear** (as in the SFR example above: the answer involves p^* but not p).
This comes out of the mathematics of the Risk Neutral Formula, which replaces $GBM(\mu, \sigma)$ with $GBM(r, \sigma)$. In passing, we pass to a “risk-neutral world”, where all stocks pay the riskless rate of return r .
2. Note the importance of the volatility parameter σ in the formula. This parameter occurs in the B-S formula, but we do not know it... So how do we deal with this?

We have to ***estimate*** it from data. How do we do that?

- a) *Historic volatility*: we can look at data back over time (*time series* data) and estimate σ from that.
- b) *Implied volatility*: we can look instead at the prices at which options are being traded today. Option prices are increasing in σ , so if we assume that the market is trading options according to the B-S formula, we can infer what value of σ the market uses.

3. *Options like volatilities*: from the explicit B-S formula above we can check by calculus that the B-S price is **strictly increasing** in the volatility σ (a little tricky to show).
4. What we need to assume to work with B-S:
 - a) No arbitrage,
 - b) Rational agents (prefer more to less – will not pass out a chance to make a riskless profit),
 - c) Markets are complete (*i.e.* Options can be replicated, in terms of a suitable combination of stock and cash – which can obviously be priced).
5. No one believes that the B-S formula holds exactly! It provides a useful first approximation (a benchmark).

9. The Black-Scholes PDE

Black and Sholes obtained their formula by showing that the price of the option satisfies the following PDE

$$f_t + rsf_s + \frac{1}{2}\sigma^2 s^2 f_{ss} - rf = 0$$

together with the terminal condition: $f(s, T) = (s - K)_+$

It is a parabolic PDE – a variant of the *heat equation*, whose solution is known!

The link between their PDE approach and our SDE approach holds more generally, as the *Feynman-Kac formula*.

10. American options: *finite time horizon*

We turn over to the case of American option with finite time-horizon T .

a) Discrete case

- i. We calculate the payoff at each terminal node, if we get to the expiry time without exercising the option before.
- ii. We work back in time one step and at each node compare the value if we exercise and if we do not (don't forget to discount): take the higher value.

- iii. We then work one more time step back and do the same as previously. We do this until we reach the root of the tree at time $t=0$.

This calculation gives two things:

- 1) the **price** of the American option – that is the value at the root of the tree.
- 2) a decomposition of the nodes into the ***exercise region*** where we exercise and the ***continuation region*** where we don't.

Here we do not have an explicit formula like in the B-S case.

However, there is some relevant theory. Consider a process $Z = (Z_n)$ with times $n = 0, 1, \dots, N$, write $U = (U_n)$ for the process given by:

$$U_N := Z_N;$$

$$U_n := \max(Z_n, E(U_{n+1} | \mathcal{F}_n)) \quad (n \leq N - 1)$$

The process U is called the *Snell envelope* of the process Z . This process is of backward induction (or dynamic programming): by working backward through the tree produces the Snell envelope (under the EMM P^* and its expectation E^*) of the value process of the European option.

The Snell envelope solves the optimal stopping problem for the American option: it maximises the expected value of our payoff conditional on present information only.

b) Continuous case

Again we use the Snell envelope.

The options seen above - European or American, put or call – are standard now: referred to as *vanilla options*.

There are other - *exotic options* (Asian options, barrier options, etc.).

All these are options on stock but there are also options on the money Markets.

The foreign exchange (forex) markets give rise to options to swap one currency for another (*swaptions*)